§ 5.4 Mean value theorem and applications

In the following we shall look at differentiable functions on the interval $\Omega = (a,b) \subset \mathbb{R}$.

Proposition 5.9 (Mean value theorem):

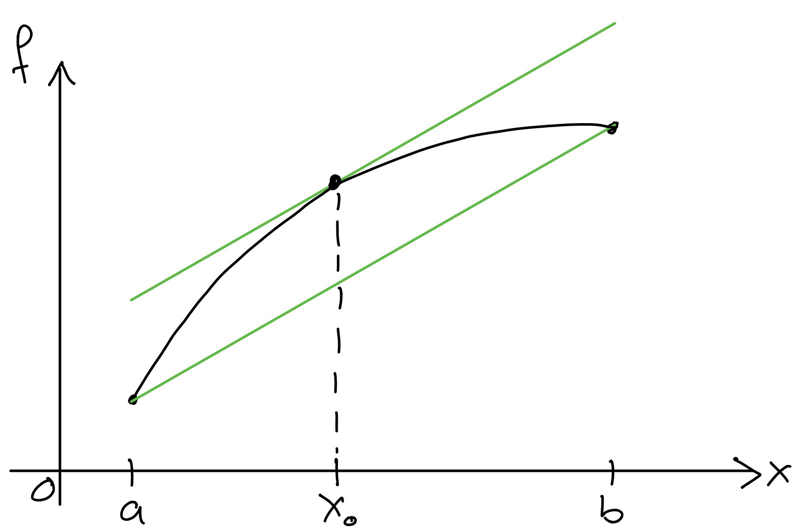
Let $-\infty < a < b < \infty$. Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous and differentiable on (a,b). Then there exists $x_0 \in (a,b)$ with

$$f(b) = f(a) + f'(x_0)(b-a);$$

that is,

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

is the slope of the secant through $(a, f(a))$, $(b, f(b)) \in \mathcal{G}(f)$



Proof:

i) We first look at the case where $f(a) = f(b) = 0$.

Then, since f is continuous, there exist $\underline{x}, \overline{x} \in [a,b]$ such that

$$f(\underline{x}) = \min_{a \le x \le b} f(x) \le 0$$

and $$f(\overline{x}) = \max_{a \le x \le b} f(x) \ge 0$$

(recall that $f([a,b]) = J \subset \mathbb{R}$ is an interval)

If $f(\underline{x}) = 0 = f(\overline{x})$, then $f = 0$; then also $f'(x) = 0 \ \forall \ x \in (a,b)$.

Otherwise let without loss of generality $f(\overline{x}) > 0$

Then $a < \overline{x} < b$, and we have

$$0 \ge \lim_{x \searrow \overline{x}} \frac{f(x) - f(\overline{x})}{x - \overline{x}} = f'(\overline{x}) = \lim_{x \nearrow \overline{x}} \frac{f(x) - f(\overline{x})}{x - \overline{x}} \ge 0,$$

so $f'(\overline{x}) = 0$.

ii) For general f consider the function $g : [a,b] \longrightarrow \mathbb{R}$ with

$$g(x) = f(x) - \left(f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right).$$

Aparently, g is continuous on $[a,b]$ and differentiable on (a,b) with

$$g'(x) = f'(x) - \frac{f(b)-f(a)}{a-b}, \quad x \in (a,b).$$

Furthermare, $g(a) = 0 = g(b)$. i) $\Rightarrow$ claim $\square$

As a first application we have

<u>Corollary 5.1:</u>

Let f be as in Prop. 5.9.
If $f' = 0$ on (a,b), then f is constant.

<u>Proof:</u>

For $a \le x < y \le b$ there exists $x_0 \in (x,y)$ with

$$\frac{f(y)-f(x)}{y-x} = f'(x_0) = 0$$

$\Rightarrow$ claim follows. $\square$

<u>Proposition 5.10:</u>

Let $f: [a,b] \longrightarrow \mathbb{R}$ be continuous and differentiable on (a,b).

i) If $\forall\ x \in (a,b)$ $f'(x) \ge 0$ (or $f'(x) > 0$, $f'(x) \le 0$, $f'(x) < 0$), then f is monotonically increasing

(or strictly monotonically increasing,
monotonically decreasing,
strictly monotonically decreasing)
on $[a, b]$.

ii) Is f monotonically increasing (or decreasing),
then we have $f'(x) \geq 0$ (or $f'(x) \leq 0$)
for all $x \in (a, b)$.

Proof:

i) We just look at the case $f'(x) > 0 \ \forall \ x \in (a, b)$
(the other cases are analogous).
Assume that f is not strictly monotonically increasing. Then $\exists \ x_1, x_2 \in [a, b]$ with $x_1 < x_2$ and $f(x_1) \geq f(x_2)$. Prop. 5.9 $\Rightarrow \exists \ x_0 \in [x_1, x_2]$ s.t.

$$f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0.$$

This is a contradiction to $f'(x_0) > 0$.
$\Rightarrow f$ is strictly monotonically increasing.

ii) Let f be monotonically increasing.
Then we have for all $x, x_0 \in (x_1, x_2)$, $x \neq x_0$:

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Taking the limit $x \to x_0$ then gives $f'(x_0) \geq 0$.

$\square$

Remark 5.4:

Is f strictly monotonically increasing, then it does not automatically follow that $f'(x) > 0$ for all $x \in (x_1, x_2)$. This is shown by the example of the strictly monotonic function $f(x) = x^3$ for which we have $f'(0) = 0$.

Example 5.13:

i) For a differentiable function $f: \mathbb{R} \to \mathbb{R}$ and a $\lambda \in \mathbb{R}$ we have $f' = \lambda f$, so

$$\forall \ t \in \mathbb{R}: \ f'(t) = \lambda f(t)$$

Then

$$\forall \ t \in \mathbb{R}: \ f(t) = f(0) e^{\lambda t}$$

Proof:

Consider the differentiable function g with

$$g(t) = \frac{f(t)}{e^{\lambda t}} = e^{-\lambda t} f(t), \quad t \in \mathbb{R}$$

Then we have

$$g'(t) = \frac{d}{dt}(e^{-\lambda t}) \cdot f(t) + e^{-\lambda t} f'(t)$$

$$= e^{-\lambda t}\left(-\lambda f(t) + f'(t)\right) = 0$$

for all $t \in \mathbb{R}$; so

$$g(t) = g(0) = f(0)$$

due to $e^0 = 1$ and Corollary 5.1,

and $\forall\, t \in \mathbb{R}: \ f(t) = e^{\lambda t} g(t) = f(0) e^{\lambda t}$.

$\square$

ii) The function $f: x \longmapsto \frac{2x}{1+x^2}$ satisfies

$$f'(x) = \frac{2(1+x^2) - 4x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2} < 0$$

for $x > 1$; thus $f: (1,\infty) \longrightarrow (0,1)$ is strictly monotonically decreasing.

Corollary 5.2 (l'Hospital rule):

Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be continuous and differentiable on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$. Further, let $f(a) = 0 = g(a)$, and let $\lim_{x \downarrow a} \frac{f'(x)}{g'(x)} =: A$.

Then $g(x) \neq 0$ for all $x > a$, and we have

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = A .$$

Proof:

i) If $g(x) = 0$ for some $x > a$, then there exists according to Prop. 5.9 a $x_0 \in (a, x)$ with $g'(x_0) = 0$ in contradiction to our assumption.

ii) For fixed $s > a$ consider the function

$$h(x) = \frac{f(s)}{g(s)} g(x) - f(x), \quad x \in [a, s] .$$

The function $h : [a, s] \longrightarrow \mathbb{R}$ is continuous and differentiable on (a, s) with $h(a) = 0$ and $h(s) = 0$. According to Prop. 5.9 there exists an $x = x(s) \in (a, s)$ such that

$$0 = h'(x) = \frac{f(s)}{g(s)} g'(x) - f'(x) ;$$

That is, $\frac{f(s)}{g(s)} = \frac{f'(x)}{g'(x)}$. With $s \longrightarrow a$ we then have $x(s) \longrightarrow a$, and

$$\frac{f(s)}{g(s)} \longrightarrow \lim_{x \downarrow a} \frac{f'(x)}{g'(x)} = A . \qquad \square$$

Example 5.14:

i) We have

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{2x}{1} = 2.$$

ii) We have

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1.$$

iii) We can also apply the l'Hospital rule several times. With ii) we get

$$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2} = \lim_{x \to 0} \frac{\sin(x)}{2x} = \frac{1}{2}.$$

However, in many cases we don't need the l'Hospital rule and can reach our conclusions with other methods:

iv)

$$\lim_{x \downarrow 0} \left(\frac{e^{-\frac{1}{x}}}{x^k} \right) \overset{y := \frac{1}{x}}{=} \lim_{y \to \infty} \left(y^k e^{-y} \right) = 0$$